MAT 1322
Lecture

Power series

03/06/2017

Power series are special series, which contain a variable x.

$$\sum_{n=0}^{\infty} \underset{\substack{\uparrow \\ \text{coefficient}}}{C_n} \cdot \underset{\substack{\nwarrow \\ \text{variable}}}{x^n}$$

Special case all $C_n = 1$ $\quad \sum_{n=0}^{\infty} 1 \cdot x^n = \sum_{n=0}^{\infty} x^n$ * geometric series

This is now a geometric series which converges exactly if $|x| < 1$.

A power series can converge for all x or just some values of x and diverge for others.

Ex $\sum_{n=1}^{\infty} \frac{2^n}{n!}(4x-8)^n \rightarrow$ contains x^n (raised to a power), so it's a power series

For which values of x does this converge?

factorial $\xrightarrow{\text{use}}$ ratio test

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$, if $L < 1$, it converges

$$a_{n+1} = \frac{2^{n+1}}{(n+1)!}(4x-8)^{n+1} \qquad (n+1)! = n!(n+1)$$

$$\lim_{n\to\infty} \frac{\left| \frac{2^{n+1}}{(n+1)!}(4x-8)^{n+1} \right|}{\left| \frac{2^n}{n!}(4x-8)^n \right|}$$

$$\lim_{n\to\infty} \left| \frac{\frac{2}{n!(n+1)}(4x-8)}{\frac{1}{n!}} \right| = \lim_{n\to\infty} \left| \frac{2}{n+1}(4x-8) \right| = 0 \quad \text{for all } x!$$

This means that the power series converges for all values of $x \in \mathbb{R}$.

Ex 2 $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$ root test: 2^n, and NO $n!$ here

$\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ for convergence.

$$L = \lim_{n\to\infty} \sqrt[n]{\left|\frac{2^n}{n}(4x-8)^n\right|}$$

$$= \lim_{n\to\infty} \frac{2}{\underbrace{\sqrt[n]{n}}_{\to 1}} |4x-8|$$

$$= 2\cdot|4x-8| = |8x-16| < 1 \text{ for convergence.}$$

This is an inequality w/ an absolute value: 2 cases!

case 1: $8x-16<1$

$8x < 17$

$\boxed{x < 17/8}$

case 2: $-(8x-16) < 1$

$-8x+16 < 1$

$-8x < -15$

$\boxed{x > 15/8}$ flip < to >

The power series for $15/8 < x < 17/8$; this is the interval of convergence

~~Rac~~ Radius of convergence:

- half the length of the interval of convergence: $R = \left(\frac{17}{8} - \frac{15}{8}\right)\cdot\frac{1}{2} = \frac{2}{8}\cdot\frac{1}{2} \boxed{= \frac{1}{8}}$

If the limit L of the root test is 1, we get no conclusion.

In the example, this is the case if $x = 15/8$ or $x = 17/8$.

So we need to check separately if the series converges or diverges for the edgepoints of the interval.

case 1 $x = 15/8$

$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(4\cdot\frac{15}{8} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{-1}{2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{-2}{2}\right)^n \cdot \frac{1}{n} \boxed{= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}}$$

* alternating harmonic series

which converges

The power series converges for $x = 15/8$, so we include $15/8$ in the interval of convergence.

case 2 $x = 17/8$

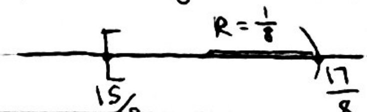
$$\sum_{n=1}^{\infty} \frac{2^n}{n}\left(4\cdot\frac{17}{8} - 8\right)^n = \sum_{n=1}^{\infty} \frac{2^n}{n}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n}$$

* harmonic series

↳ diverges

CONCLUSION: $x = 17/8$ is not in the interval of convergence.

Interval of convergence: $\frac{15}{8} \le x < \frac{17}{8}$



Theorem There are 3 cases for $\sum_{n=0}^{\infty} c_n (x-a)^n$:

(i) converges for all x (ex 1)
(ii) converges only if $x = a$
(iii) converges if $|x-a| < R$ and diverges if $|x-a| > R$
it can converge or diverge if $|x-a| = R$ (ex 2)
R is called the radius of convergence

Ex 3 $\sum_{n=1}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$ find the interval and radius of convergence.

<u>root test</u>: $\lim_{n\to\infty} \sqrt[n]{\left|\frac{n(x+2)^n}{3^{n+1}}\right|} = \lim_{n\to\infty} \frac{\overset{\to 1}{\sqrt[n]{n}}}{\underset{\to 1}{\sqrt[n]{3}}\cdot 3} |x+2|$

$= \lim_{n\to\infty} \frac{1}{3}|x+2| = \frac{1}{3}|x+2| < 1$ for convergence

$\frac{1}{3}|x+2| < 1 \Rightarrow |x+2| < 3$ ← radius of convergence!

<u>case 1</u> $x+2 < 3$
$\underline{x < 1}$

<u>case 2</u> $-(x+2) < 3$
$-x < 5$
$\underline{x > 5}$

* check convergence at the bounds!

$x=1$ $\sum_{n=1}^{\infty} \frac{n\cdot 3^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{n\cdot 3^n}{3^n\cdot 3} = \sum_{n=1}^{\infty} \frac{n}{3}$ } diverges $\lim_{n\to\infty} \frac{n}{3} = \infty$, not 0

$x=-5$ $\sum_{n=1}^{\infty} \frac{n\cdot(-3)^n}{3^n\cdot 3} = \sum_{n=1}^{\infty} \frac{n\cdot 3^n(-1)^n}{3^n\cdot 3} = \sum_{n=1}^{\infty} \frac{n\cdot(-1)^n}{3}$ } diverges

$\left(\sum_{n=1}^{\infty} \frac{n(-1)^n}{3} = \frac{1}{3}(\underbrace{-1+2}_{1}\underbrace{-3+4}_{1}\underbrace{-5+6}_{1}\underbrace{-7+8}_{1}\ldots = \infty\right)$ or, can take the Alternate Series Test

CONCLUSION: interval of convergence: $-5 < x < 1$
radius of convergence: $R = 3$ $\left(= \left(\frac{1-(-5)}{2}\right)\right)$

Power series as functions

Ex $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ this is a function $f(x) = \frac{1}{1-x}$

So, power series can describe functions!

We want to integrate $\frac{1}{1-x}$, $\int \frac{1}{1-x}\,dx = ?$ → we could use calc 1, or:

$$\int \left(\sum_{n=0}^{\infty} x^n\right) dx = \sum_{n=0}^{\infty} \left(\int x^n\,dx\right)$$

$$= \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} + C$$

CALC I $\int \frac{1}{1-x}\,dx = \left| \begin{array}{l} u = 1-x \\ du = -dx \end{array} \right| = -\int \frac{1}{u}\,du$

$$= -\ln|u| = -\ln|1-x| + C$$

SO:

$$-\ln|1-x| = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}.$$